CALCULATING MAX-EIGENVALUE AND MAX-EIGENVECTOR WITH JUMPS OF MATRICES

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ABSTRACT. The eigenvector problem for an irreducible non negative matrix $A = [a_{ij}]$ in the max algebras is the form $A \otimes x = \lambda x$, where $(A \otimes x)_i = max(a_{ij}x_j)$, $x = (x_1, x_2, \dots, x_n)^T$ and λ refers to the maximum cycle geometric mean $\mu(A)$. In this paper, we exhibit a method to compute $\mu(A)$ and max-eigenvector by using mutations of matrices. Since the order of power method algorithm is $O(n^3)$, the advantage of this paper present a faster procedure.

1. Introduction

The *max-algebra* or $(\max, \times)$ semi ring, is the set $\mathbb{R}_+$, equipped with max as addition, and ordinary multiplication as its multiplication. It is traditional to use the notation $\oplus$ for max, and $\otimes$ for $\times$. For vectors $x = (x_i), y = (y_i)$ in $\mathbb{R}_+^n$ and $c \in \mathbb{R}_+$ the vectors $x \oplus y = (\max\{x_i, y_i\})$ and $cx = (cx_i)$ are defined element wise. The sum $A \oplus B$ of two matrices is defined analogously. This structure satisfies all the semi ring axioms, i.e. $\oplus$ is associative, commutative, with zero element, $\otimes$ is associative, has a unit, distributes over $\oplus$, and zero is absorbing. This semi ring is commutative $(a \oplus b = b \oplus a)$, idempotent $(a \oplus a = a)$, and non zero elements have an inverse for $\otimes$. The term diode is sometimes used for an idempotent semi ring. Using the new symbols $\oplus$ and $\otimes$ instead of the familiar max and $\times$ notation is the price to pay to easily handle all the familiar algebraic constructions. For instance, we will write, in the Max semi ring:

$$\begin{bmatrix} 2 & 0 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \end{bmatrix} = \begin{bmatrix} 2 \otimes 10 \oplus 0 \otimes 13 \\ 11 \otimes 10 \oplus 15 \otimes 13 \end{bmatrix} = \begin{bmatrix} 20 \\ 195 \end{bmatrix}, \tag{1}$$

$$(3 + x)^2 = (3 + x)(3 + x) = 6 \oplus 3x \oplus x^2 = 6 \oplus x^2. \tag{2}$$

Obviously with this multiplication for matrices, we can define max eigenvalues and eigenvectors of a matrix. If for some matrix A a vector $x \geq 0$ and a number $\mu(A)$ exist such that

$$A \otimes x = \mu(A)x, \quad (A \otimes x)_i = \max_j a_{ij}x_j, \tag{3}$$

Then x is called an max eigenvector of A, and $\mu(A)$ an max eigenvalue of A. For abbreviation we simply use eigenvalue and eigenvector instead of max eigenvalue and eigenvector.

There are several applications of max algebras one can found in [1, 2, 5, 10, 14]. As it clearly showed in application, max algebras and especially eigenvalues and eigenvectors can describe some important properties.

One of important application of max algebras is discrete events dynamical systems (DEDS). In control engineering, a discrete event dynamic system (DEDS) is a discrete-state, event-driven system of which the state evolution depends entirely on the occurrence of asynchronous discrete events over time. Although similar

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to continuous-variable dynamic systems (CVDS), DEDS consists solely of discrete state spaces and event-driven state transition mechanisms. We can correspond to the graph of every such systems, a matrix A which every entry A_{ij} is the weight of arc (j, i) [4].

A path of length l in a graph is a sequence of arcs

$$(i_1, i_2), (i_2, i_3), \dots, (i_l, i_{l+1}). \tag{4}$$

A *circuit* of length l is a closed path of length l. A circuit which its first and last elements coincide is called a *Cycle*.

Definition 1.1. A graph is called *strongly connected* if from any node i to any node j a path exists. If a graph is strongly connected, the corresponding matrix will be called *irreducible*. A non negative matrix $A \in M_n$ is said to be *primitive* if it is irreducible and has only one eigenvalue of maximum modules.

Irreducible matrices have a special matrix form. Indeed, a reducible matrix $A \in M_n$ of size 1 is 0 and for $n \geq 2$, there is a perjump matrix $P \in M_n$ and an integer r with $1 \leq r \leq n-1$ such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{5}$$

Where $B \in M_r$, $D \in M_{n-r}$, $C \in M_{r,n-r}$ and $0 \in M_{n-r,r}$ is a zero matrix. A matrix A is irreducible if it is not reducible.

Definition 1.2. The *weight* of a path $i_1 \to i_2 \to \dots i_l \to i_{l+1}$ is the sum of the weights of the individual arcs. The *average weight* of a path is its weight divided by the number of arcs: $(A_{i_2,i_1} + A_{i_3,i_2} + \dots + A_{i_l,i_{l-1}})/(l-1)$. The *circuit mean* is the average weight of a circuit. Any circuit of maximum average weight is called a *critical circuit*. Similarly, we can define *circuit geometrical mean* by

$$\sqrt[n]{A_{i_2,i_1} \times A_{i_3,i_2} \times \dots \times A_{i_l,i_{l-1}}}. \tag{6}$$

Definition 1.3. A matrix $A \in M_n$ is said to have *property SC* if for every pair of distinct integers p, q with $1 < p, q < n$ there is a sequence of distinct integers $k_1 = p, k_2, k_3, \dots, k_{m-1}, k_m = q$, $1 < m < n$, such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, \dots, a_{k_{m-1}k_m}$ are non zero.

Several related subjects about property SC can be found in [12]. We just mention to following theorem without proof.

Theorem 1.4. *Let $A \in M_n$ corresponding to graph $\Gamma(A)$. The following are equivalent:*

(a) *A is irreducible;*
(b) *$\Gamma(A)$ is strongly connected;*
(c) *A has property SC.*

Braker and Olsder present in [4] a power algorithm to find max eigenvalue. They also found some useful properties. Esner and Van Den Driessche try to modify Braker's method. In section 2, we try to present a method, based on concept of jump of matrices which is faster and has smaller absolute value of error.
In section 3, we present a new method, based on max algebra, corresponding an optimization problem to a SR matrix such that minimize following function:

$$e(w) = \max |\frac{a_{ik} - w_i/w_k}{a_{ik}}|. \tag{7}$$

We also, introduce jumps of a matrix and its applications in symmetric, reciprocal and trasitive matrices and in the third section, we calculate the error refered to jumps and study behavior of jumps of transitive matrices.

2. Main Theorems

To evaluate the determinant of a matrix $A \in M_n$ by Laplace expansion, we use the following formula:

$$det\ A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\ det\ A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\ det\ A_{ij}, \tag{8}$$

Where A_{ij} is a matrix which obtain by eliminating the $i-$th row and $j-$th column of matrix A. This process is continued to evaluate $det\ A_{ij}$ while the size of remain matrix becomes equal to 2. To evaluate the determinant of a matrix with size 2, we simply differ the multiplication of entries on main diagonal and other diagonal. In fact in this process, we multiply on entry of every row and column. Determinant of the matrix is a special kind of linear combination of these multiplications.
We can have an other formula to exhibit determinant of matrix $A \in M_n$.

$$det\ A = \sum_{\sigma} sgn\ \sigma \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{9}$$

where the sum runs over all $n!$ perjumps a of the n items $\{1, \dots, n\}$ and the "sign" or "signum" of a perjump σ, $sgn\ \sigma$, is $+1$ or -1, according to whether the minimum number of transpositions, or pair wise interchanges, necessary to achieve it starting from $\{1, 2, \dots, n\}$ is even or odd. Thus, each product $a_{1\sigma(1)}a_{2\sigma(2)} \dots a_{n\sigma(n)}$ enters into the determinant with a $+$ sign if the perjump σ is even or a $-$ sign if it is odd.

Definition 2.1. For matrix $A \in M_n$, every product of n entries of different rows and columns is called a *Jump*. A Jump with no main diagonal entry as its multipliers is called a *Principal Jump*. Other wise, we called it a *Subordinate Jump*.

In this view, determinant of a matrix is a special linear combination of its Jumps.

Example 2.2. Consider the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}. \tag{10}$$

We have

$$det\ A = 1 \times 5 \times 9 - 1 \times 6 \times 8 + 2 \times 4 \times 9 - 2 \times 6 \times 7 + 3 \times 4 \times 8 - 3 \times 5 \times 7. \tag{11}$$

In this example, $3\times4\times8$ and $2\times6\times7$ are principal Jumps and others are subordinate Jumps.

Theorem 2.3. *Let A be a $n \times n$ non negative matrix. The part of jump that hasn't any entry of principal diagonal form a cycle.*

Proof. Let α is a Jumps of A. Then entries of α are $a_{i_1i_2}, a_{i_2i_3}, \dots, a_{i_ki_1}$ which diagonal entries of A are eliminated. By definition, α is a cycle. □

Theorem 2.4. *Let A be a $n \times n$ non negative matrix. A part of a Jumps with non zero production of elements and without any element of a principal Jump form a cycle.*

Proof. It is not hard to show, using lemma 2. □

Definition 2.5. Let A be a square matrix. The set of all Jumps of A is denoted by $M(A)$. Let α be a Jump of A. The product of non zero elements of α is denoted by $p(\alpha)$ and the number of non zero elements of α is denoted by S_α.

Theorem 2.6. *Let A be a non negative square matrix and $\alpha \in M(A)$. Then*

$$\mu(A) = \max\{ \sqrt[S_\alpha]{p(\alpha)}\ ,\ \alpha \in M(A)\}. \tag{12}$$

Proof. By using of two primer theorem and knowing that a circuit can appear on a vertex therefore element of principal diagonal can yield a circuit then whit attention to define of $\mu(A)$ proof is evident. □

Example 2.7. The eigenvalue problem for an irreducible non negative matrix $A = [a_{ij}]$ in the max algebras is

$$A \otimes x = \lambda x \quad (A \otimes \lambda)_i = \max_j a_{ij} x_j, \tag{13}$$

Where λ denotes the maximum cycle mean $\mu(A)$. But by properties of max-algebras

$$A^* = [a_{ij}] = \begin{cases} a_{ij} & \textit{if } (i,j) \textit{ is in jump } \mu(A) \\ 0 & \textit{other wise} \end{cases}, \tag{14}$$

And $A^* x = \mu(A) x$ we can simply obtain eigenvalues of A.

Example 2.8. Consider matrix A as follows:

$$A = \begin{bmatrix} 0 & 8 & 1 \\ 3 & 0 & 2 \\ 4 & 1 & 1 \end{bmatrix}. \tag{15}$$

Then

$$det(A) = 0 \cdot 0 \cdot 1 - 0 \cdot 2 \cdot 1 - 8 \cdot 3 \cdot 1 + 8 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 1 - 1 \cdot 0 \cdot 4, \tag{16}$$

$$\mu(A) = \max\{\sqrt[2]{2 \cdot 1}, \sqrt[3]{8 \cdot 2 \cdot 4}, \sqrt[3]{1 \cdot 3 \cdot 1}\} = 4. \tag{17}$$

Where $\mu(A)$ is corresponding to subordinate jump a_{12}, a_{21}, a_{33}. Thus by equation $A^* x = \mu(A) x$ we have

$$\begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T 4. \tag{18}$$

So

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix}. \tag{19}$$

Theorem 2.9. *Let A be a $n \times n$ non negative matrix. Then*

$$\prod a_{ii} > \max\{p(\alpha)\ ,\ \alpha \in M(A)\} \Longrightarrow \mu(A) = \max\{a_{ii}\}. \tag{20}$$

Proof. When Πa_{ii} is greater than all $p(\alpha)$'s, then there one of the main diagonal entry has the greatest module. Then the maximum of geometric mean holds on the main diagonal. □

Theorem 2.10. *Let A be a non negative square matrix. Then*

$$\forall \alpha \in M(A), p(\alpha) < 1 \Longleftrightarrow \mu(A) < 1.$$

Proof. It is not hard to see, since the maximum of geometric mean is equal to $S_\alpha - th$ root of some jump α. □

Theorem 2.11. *Let A be a non negative square matrix. If maximum geometric mean $\mu(A)$ holds on a jump with only elements of principal diagonal, then max-eigenvector corresponding with $\mu(A)$ is e_i, where i is the index of $\mu(A) = a_{ii}$, else $x \neq e_i$.*

Proof. By the condition of theorem, all entries of A^* is equal to zero unless the entry corresponding to $\mu(A) = a_{ii}$. Then by $A^*x = \mu(A)x$, e_i is the eigenvector corresponding to $\mu(A)$. If the maximum geometric mean $\mu(A)$ holds on any other jump, there is a non zero entry in every row of A^* since there is an entry of every row and column of A in this jump. So $A^*x = \mu(A)x$ concludes $\mu(A) > 0$. □

The following theorem shows the relation between critical matrix and jumps.

Theorem 2.12. *Let A be a non negative matrix. The critical matrix of A is corresponding with a non zero jump which holds on its Jump cyclic geometric mean $\mu(A)$, therefore*

$$a_{ij}{}^c = \begin{cases} a_{ij} & \text{if } (i,j) \text{ exist in Jump of } \mu(A) \\ 0 & \text{else} \end{cases}. \tag{21}$$

Proof. It is not hard to see by 12 and definition of critical matrix. □

3. Application of jumps in Producing Transitive Matrices

An (entrywise) positive $n \times n$ matrix $A = (a_{ij})$ is called a symmetrically reciprocal matrix (SR- matrix) if $a_{ij}a_{ji} = 1$ for all i,j=1,...,n, thus $a_{ii} = 1$. The firest time were introduced by Saaty, used in the analytic hierarchy process(AHP method) for multicriteria decision making. And An SR- matrix $B = (b_{ij})$ is called transitive if there is a positive n-vector $W = (w_1, w_2, \dots, w_n)$ such that $b_{ij} = \dfrac{w_i}{w_j}$ for $i, j = 1, \dots, n$.thus sometimes it is required to deduce positive weights $w_1, w_2, dots, w_n$ attached to the alternatives $A_1, \dots, A_n$ respectively, from the SR-matix A. Approximating an SR-matrix A by a transitive matrix B is an importance step in the analytic hierarchy process(AHP) for decision making by attaching a ranking to the SR- matrix. In this way the alternatives can be ranked. For the ideal case, $a_{ij} = \dfrac{w_i}{w_j}$ for $i, j = 1, \dots, n$. it should be noted a transitive matrix is SR-matrix of rank one. However, in a realistic case a_{ij} is only approximately given by $\dfrac{w_i}{w_j}$. one of the most important subjects in section is , costructing a weight vector $W = (w_1, w_2, \dots, w_n)$ that there are several suggstions in literature for constructing a weight vector . Some of these methods is given below. Saaty proposes taking the perron vector of A . the vector W is chosen in such a way that the matrix with entries $\dfrac{w_i}{w_j}$ has minimal distance from A in the Euclidean matix norm, i.e., that

$$\sum (a_{ik} - \frac{w_i}{w_j})^2$$

is minimal. in the for costructing a weight vector $W = (w_1, w_2, \dots, w_n)$ using that of max-eginvector of A that solves a useful optimization problem, namely minimizing the relative error useful optimization problem, namely minimizing relative error

$$e(W) = \max \left|\frac{a_{ik} - (w_i/w_j)}{a_{ik}}\right| \qquad (*)$$

but we in this peaper given new method for costructing a weight vector $W = (w_1, w_2, \dots, w_n)$ with using of jumps of matrix . This method has the following advantages compared to other methods that are Short out the answer and Makes much more efficient error (*). the firest we mention several theorem about relation between jumps and SR-marix and transitive matrix.

Theorem 3.1. *A matrix $A \in M_n (n \geq 3)$ is symmetrically reciprocal if and only if the production of all its subordinate jumps equals 1.*

Proof. Let A be a symmetrically reciprocal matrix. By definition, a subordinate jump contain some elements like a_{ij}, a_{ji} and a_{ii} and by the properties of SR matrices on such a subordinate jump, $\prod a_{ij} = 1$. Other side of the theorem is obvious. □

Theorem 3.2. *A symmetrically reciprocal matrix A is transitive if and only if the production of all its jumps equals 1.*

Proof. Let A be a transitive matrix, then there exist a wight vector $w = (w_1, w_2, \dots, w_n)$ such that $a_{ij} = w_i/w_j$. Form of principal jumps is a_{ij}, a_{jk}, a_{ki} and form of subordinated jump is a_{ij}, a_{ji}, a_{ii}. So we have

$$\frac{w_i}{w_j} \times \frac{w_j}{w_k} \times \frac{w_k}{w_i} = 1 \quad and \quad \frac{w_i}{w_j} \times \frac{w_j}{w_i} \times \frac{w_i}{w_i} = 1. \tag{22}$$

Other side of the theorem is obvious. □

Lemma 3.3. *For three integers a, b, c, following statements are equivalent:*

(a) $\frac{1}{1+c} \leq ab \leq 1 + c$,

(b) $|a - b| \leq ca$ *and* $|\frac{1}{a} - b| \leq \frac{c}{a}$.

Proof. See [12]. □

Theorem 3.4. *Let $A \in M_n(\mathbb{R}_+)$ and we have its determinant by Laplace expansion. If $\prod a_{ij}$ be the production of elements of each jump, $w = (w_1, w_2, \dots, w_n)$ be the weight vector and $B = [b_{ij}]$ is a matrix such that $b_{ij} = \frac{w_i}{w_j}$, then following statements are equivalent:*

(a) $\frac{1}{1+c} \leq a_{ij}b_{ij} \leq 1 + c$,

(b) $|a_{ij} - b_{ij}| \leq ca_{ij}$ *and* $|\frac{1}{a_{ij}} - b_{ij}| \leq \frac{c}{a_{ij}}$.

In special case, if the maximum production is of a principal jump, then

$$c = \sqrt[k]{\prod a_{ij}} - 1, \tag{23}$$

Where $\prod a_{ij}$ is the maximum product of jumps and k is the number of elements of it.

Proof. Let A be a SR matrix. By lemma 3.3, set $a = a_{ij}$ and $b = \frac{w_j}{w_i}$. So (a) and (b) are equivalent.
But by (7) we have

$$e(w) = \max |\frac{a_{ik} - w_i/w_k}{w_{ik}}| \quad \forall\ w > 0, \tag{24}$$

And for every row of A and B we have

$$1 + e(w) = \max(a_{ik}\frac{w_i}{w_k}). \tag{25}$$

So

$$(1 + e(w))^k = (\prod \max(a_{ik}\ \frac{w_i}{w_k}) = \max(\prod a_{ij}\ \prod \frac{w_i}{w_k}). \tag{26}$$

But $\max \frac{w_i}{w_k} = 1$. Therefore

$$(1 + e(w))^k = \max \prod a_{ik} \Rightarrow (1 + e(w)) = \sqrt[k]{\prod aik} \Rightarrow e(w) = \sqrt[k]{\prod aik} - 1. \tag{27}$$

□

Let A be a positive SR matrix and $\tau > 0$. We define $A^\tau = [a^\tau_{ij}]$ as follow:

$$a^\tau_{ij} = \begin{cases} a_{i_1j_1}\tau & (i_1, j_1) \text{ is the in the jump which } \mu(A) \text{ holds} \\ a_{j_1i_1}/\tau & (j_1, i_1) \text{ is the in the jump which } 1/\mu(A) \text{ holds} \\ a_{ij} & \text{other wise} \end{cases}. \tag{28}$$

Lemma 3.5. *Let A be the matrix defined in 28 with $n \geq 2$. There exist positive numbers τ_1 and τ_2 such that $\mu(\tau)$ is strictly decreasing on $(0, \tau_1)$, is constant on (τ_1, τ_2) and is strictly increasing on $(\tau_2, +\infty)$. Moreover for $\tau \in (0, \tau_1) \cup (\tau_2, +\infty)$, eigenvector is unique.*

Proof. Since $\mu(\tau)$ is constructed by elements of a principal jump and the product of principal jumps increases on interval $(-\infty, \tau_1)$, so $\mu(\tau)$ decreases on this interval and increases on (τ_2, ∞). □

Lemma 3.6. *By lemma 3.5, if we define $\mu_0 = \min\{\mu(\tau), \tau > 0\}$ then following statements hold:*

(a) $\mu_0 = 1, \tau = \prod a_{ij}$,
(b) $\tau_1 < \prod a_{ij}, \tau_2 > \prod a_{ij}$,
(c) $\tau = \emptyset$,
(d) $\tau = \tau_0$.

Proof. (a) Since A is a SR matrix and by changing the elements of jump, the product of its element changes, but this minimum equals 1 if and only if A is transitive. So

$$\mu(\tau) = \sqrt[k]{\prod a_{ij}} = 1 \Rightarrow \mu(\tau) = \mu_0 = 1. \tag{29}$$

(b) Let (i_0, j_0) be a coordinate which $a_{i_0 j_0}$ is an element of $\mu(\tau) = \sqrt[k]{\prod a_{ij}}$. Since by definition, changing the elements a_{ij} to $\frac{a_{ij}}{\tau}$ and changing the coordinates to (j, i) cause change of elements to $a_{ij}\tau$, while a_{ij} are in principal jump, so the maximum change holds when it occures on a principal jump.
(c) It is clear by definitions.
(d) If $\tau = \tau_0$ Then A is a transitive matrix with $\prod a_{ij} = 1$. So $rank(A) = 1$. □

Theorem 3.7. *Let $\mu(\tau)$ be the eigenvector defined on 28and element that changes a_{ij}, $0 < \tau_0 < \tau_1$ and x, y be two max eigenvector of $A(\tau)$ such that*

$$A(\tau_0) \otimes x = \mu(\tau)x \qquad A(\tau_1) \otimes y = \mu(\tau)y. \tag{30}$$

Then

$$\mu(\tau_0) < \mu(\tau_1) \Rightarrow \frac{y_1}{x_1} > \frac{y_i}{x_i} \quad i \geq 2 \quad \text{and} \quad \mu(\tau_0) > \mu(\tau_1) \Rightarrow \frac{y_2}{x_2} < \frac{y_i}{x_i} \quad i \neq 2. \tag{31}$$

Proof. Let A is a SR matrix and $a_{1,2}$ is changed to $a_{1,2}t$ than since $\tau_0 < \tau_1$ and by attention of propety of max algebra and by helping of matix $A(\tau_0)^*$ and $A(\tau_1)^*$ that introduced in second section we have

$$\begin{aligned} \tau_0 a_{1,2} x_2 &= \mu_0 x_1 \\ a_{ik} x_k &= \mu_0 x_i \\ &\vdots \\ a_{ps} x_s &= \mu_0 x_p \end{aligned}$$

and

$$\begin{aligned} \tau_1 a_{1,2} x_2 &= \mu_1 x_1 \\ a_{ik} x_k &= \mu_1 x_i \\ &\vdots \\ a_{ps} x_s &= \mu_1 x_p \end{aligned}$$

So we have the first equations

$$\tau_0 a_{1,2} x_2 = \mu_0 x_1 \Rightarrow \mu_0 = \tau_0 a_{1,2} \frac{x_1}{x_2}$$

and

$$\tau_1 a_{1,2} x_2 = \mu_1 x_1 \Rightarrow \mu_1 = \tau_0 a_{1,2} \frac{x_1}{x_2}$$

now since

$$\mu_0 < \mu_1 \Rightarrow \tau_0 a_{1,2} \frac{x_1}{x_2} < \tau_0 a_{1,2} \frac{y_1}{y_2}$$

therefor $\frac{x_1}{x_2} < \frac{x_1}{x_2}$ and other Equations we can easily obtain Other inequality . The second result follows from (i) by interchanging the roles of $A(\tau_0)$ and $A(\tau_1)$. □

Corollary 3.8. *With condition of Theorem* 3.7*, if* $\mu(\tau_0) = \mu(\tau_1)$ *then*

$$\frac{y_1}{x_1} \geq \frac{y_i}{x_i} \geq \frac{y_2}{x_2}. \tag{32}$$

Proof. It is a direct conclusion of previews theromes. □

References

[1] Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris Nikoletseas and Wolfgang Thomas, *Automata, Languages and Programming*, 36th International Colloquium, ICALP 2009, Rhodes, Greece, July 5-12, 2009, Proceedings, Part I, Springer- Verlag Berlin Heidelberg 2009.

[2] F.L. Baccelli, G. Cohen, G.J. Olsder and J.-P. Quadrat, *synchronization and Linearity: An Alebra for Discret Event System*, Wiley, New York,1992.

[3] R.B. Bapat, *A max version of the Perron-Frobenius theorem*, Linear Algebra Appl. 275-276 (1998) 3-18.

[4] J. G. Braker and G. J. Olsder, *The Power Algorithm in Max Algebra*, Linear Algebra and Its Application 182: 67-89 (1993).

[5] Peter Butkovic, *Max-linear Systems: Theory and Algorithms*, Springer-Verlag London Limited, 2010.

[6] L. Esner and P. Van Driessche, *On the power method in max-algebras*, Linear Algebra Appl. 302-303 (1999) 17-32.

[7] L. Esner and P. Van Driessche, *Modifying the method in max-algebras*, Linear Algebra Appl. 332-385 (2001)3-13.

[8] L. Esner and P. Van Driessche, *Max-algebra and pairwise comparison matrices*, Linear Algebra Appl. 385 (2004) 47-62.

[9] L. Elsner and van den Driessche, *Max-algebra and pair wise comparison matrices, II*, Linear Algebra and its Applications 432 (2010) 927935.

[10] Stephane Gaubert, *Methods and Applications of (max,+) Linear Algebra*, HAL Id: inria-00073603 https://hal.inria.fr/inria-00073603.

[11] Bernd Heidergott, *Max-Plus Linear Stochastic Systems and Perturbation Analysis,*, Springer Science+Business Media, LLC, 2006.

[12] Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, Cambridge University Press , 1985.

[13] Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analsis*, Combridge Universty Press, Cambridge, 1999.

[14] William M. McEneaney, *Max-plus methods for nonlinear control and estimation*, Birkhauser Boston, 2006.

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